

NONORIENTABLE FOUR-BALL GENUS CAN BE ARBITRARILY LARGE

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ABSTRACT. The nonorientable four-ball genus $\gamma_4(K)$ of a knot $K \subset S^3$ is the smallest first Betti number of any smoothly embedded, nonorientable surface $F \subset B^4$ bounding K . In contrast to the orientable four-ball genus, which is bounded below by the invariants σ , τ , and s , the best lower bound in the literature on $\gamma_4(K)$ for any K is 3. We prove that

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - d(S^3_{-1}(K)),$$

where the first term is half the knot signature, and the second is the Heegaard-Floer d -invariant of the integer homology sphere given by -1 surgery on K . In particular, we show that $\gamma_4(T_{2k,2k-1}) = k - 1$.

1. INTRODUCTION

One measure of the complexity of a knot $K \subset S^3$ is the complexity, as codified by genus, of the simplest surface which bounds it. For example, the only knot which bounds a genus zero surface embedded in S^3 is the unknot. This definition of complexity depends dramatically on the class of surfaces allowed: orientable or nonorientable, embedded in S^3 or B^4 , and for surfaces in B^4 , whether or not the embedding is smooth or locally flat. (The genus of a nonorientable surface with boundary is defined to be its first Betti number b_1 .) For example, the nonalternating knot 11^2_{31} from Thistlethwaite's table bounds an orientable surface of genus 3 in S^3 , a smooth orientable surface of genus 2 in B^4 , and a locally flat orientable surface of genus 1 in B^4 . Certifying the minimality of these surfaces requires a variety of modern and classical knot invariants: the Alexander polynomial Δ has degree 3, Ozsvath-Szabo's τ -invariant is equal to 2, and the Murasugi signature σ is equal to 1; to construct the final surface, Stominiew found a genus one concordance to a knot with Alexander polynomial 1, which according to a result of Freedman bounds a locally flat disk.¹

We know that orientable techniques cannot apply verbatim to obstruct nonorientable surfaces because of a simple example: the $2k + 1$ -twist torus knot $T_{2,2k+1}$ bounds a Mobius band in S^3 , yet the genus k Seifert surface in Figure 1.1 actually has minimal genus even among orientable, locally flat surfaces embedded in B^4 bounding the knot. The global property of orientability, perhaps recast as the existence of a top homology class or a complex structure, is somehow critical to both the proof and truth of the bounds involving Δ , τ , and σ . While some obstructions have been found to particular knots bounding Mobius bands or punctured Klein bottles in B^4 ([Yas, MY] see [GL1] especially for a comprehensive survey), the following question remained open:

Question. Does every knot K bound a punctured $\#^3\mathbb{RP}^2$ smoothly embedded in B^4 ?²

The answer is, perhaps unsurprisingly, “no.”

¹<http://www.indiana.edu/~knotinfo/>

²Gilmer and Livingston [GL1] use Casson-Gordon invariants to construct a family of knots K_n such that K_n does not bound a nonorientable *ribbon* surface in B^4 of genus less than n .

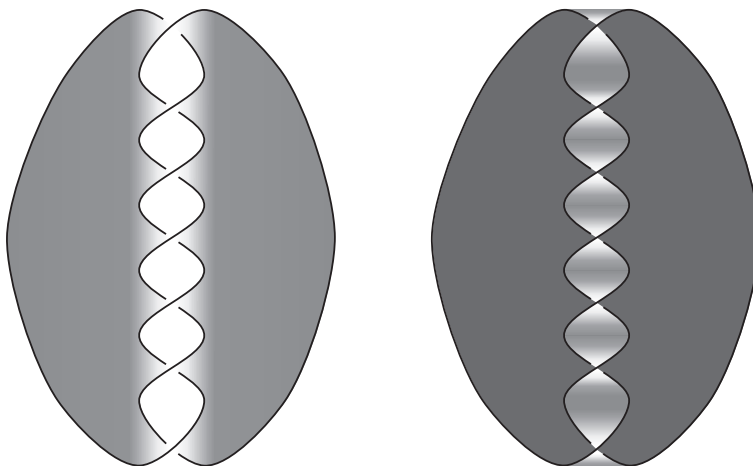


FIGURE 1.1. An orientable surface with $b_1 = k$ and a nonorientable surface with $b_1 = 1$, each bounding $T_{2,2k+1}$. Here k is 3.

Theorem 1. Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then

$$b_1(F) \geq \frac{\sigma(K)}{2} - d(S^3_{-1}(K)),$$

where σ denotes the Murasugi signature and d the Heegaard-Floer d -invariant of the integer homology sphere given by -1 surgery on K .

In particular, we show

Theorem 2. Any smoothly embedded surface $F \subset B^4$ bounding the torus knot $T_{2k,2k-1}$ has $b_1(F) \geq k - 1$.

The equivalent question in the topological category remains open.

A moment for notation: the minimal genus of any surface bounding $K \subset S^3$ will be written $g_3(K)$, $g_4(K)$, $g_4^{top}(K)$, $\gamma_3(K)$, $\gamma_4(K)$, $\gamma_4^{top}(K)$ depending on whether we allow orientable or nonorientable surfaces (g vs. γ) embedded in S^3 or B^4 (3 vs. 4) smoothly or topologically (no superscript vs. *top*). Thus for $K = 11^2_{31}$, we have

$$2g_3(K) = 6 \quad 2g(K) = 4 \quad 2g_4^{top}(K) = 2 \quad \gamma_3(K) = 3 \quad \gamma_4(K) = ? \quad \gamma_4^{top}(K) = ?$$

This definite value for $\gamma_3(K)$, also called the *crosscap genus*, is due to Burton and Ozlen, who use integer programming and normal surface theory to construct a triangulation of the knot complement and certify minimal surfaces in it. Geometric techniques can also be used to exactly compute the nonorientable 3-genus of a torus knot $T_{p,q}$ —it turns out that $\gamma_3(K)$, much like $\sigma(K)$, is a recursive, arithmetic function of p and q [Ter]. Algebraic or polynomial invariants bounding γ_3 have yet to be found—the 4-dimensional results of this paper provide the only general 3-dimensional bounds known to the author.

The first large lower bounds on $g_4(K)$ are due to Murasugi, who proved that $2g_4(K) \geq \sigma(K)$. The failure of this inequality if we replace $2g$ with γ is illuminating, so we quickly recall a proof of the signature bound due to Gordon and Litherland. Let $(F, \partial F) \hookrightarrow (B^4, S^3)$ be an embedded surface bounding a knot K . The normal bundle $\nu(F)$ always admits a nonvanishing section s . On the boundary, $s|_{\partial F}$ provides a framing of K , which we use to define the *normal Euler number* of F :

$$e(F) := -\text{lk}(K, s(K)).$$

Gluing an orientable Seifert surface Σ for K to F gives a closed surface in B^4 with self-intersection $e(F)$. If F is orientable, then $F \cup \Sigma$ represents an integral homology class and self-intersection can be computed algebro-topologically; since $H_2(B^4; \mathbb{Z}) \cong 0$, $e(F)$ must be zero. If F is nonorientable, then we must compute self-intersection geometrically. Take a transverse pushoff of F , and choose arbitrary orientations in the neighborhood of each intersection point. Together with the orientation of B^4 , this allows us to assign signs to each intersection; the sum turns out to be independent of the choice of pushoff and local orientation. It must be even, since we may compute self-intersection algebraically over $\mathbb{Z}/2$, but it needn't be zero. For example, the Mobius band bounding $T_{2,n}$ has normal Euler number $-2n$.

Let $W(F)$ denote the double cover of B^4 branched over F . Gordon and Litherland [GL2] use the G -signature theorem to show that the quantity

$$\sigma(W(F)) + \frac{e(F)}{2}$$

is independent of the choice of surface F bounding K , and equal to the knot signature $\sigma(K)$. For any such F , then,

$$\left| \sigma(K) - \frac{e(F)}{2} \right| = |\sigma(W(F))| \leq b_2(W(F)) = b_1(F),$$

where the final equality can be proved using elementary algebraic topology.

This inequality is tight for both of the surfaces bounding $T_{2,2k+1}$ in Figure 1.1. The Seifert surface has $e(F) = 0$ and $b_1(F) = 2k$, the Mobius band has $e(F) = -2(2k+1)$ and $b_1(F) = 1$, and $\sigma(T_{2k,2k+1}) = -2k$. In light of the important role played by $e(F)$, it may be clarifying to sort surfaces based on the framing they induce on the knot, and try to compute

$$\gamma_4(K, n) := \min \{ \gamma(F) \mid (F, \partial F) \hookrightarrow (B^4, K) \text{ and } e(F) = 2n \}.$$

The signature inequality, in this notation, is $\gamma_4(K, n) \geq |\sigma(K) - n|$.

The strategy of this paper is as follows. First, we replace our nonorientable surface in B^4 with an orientable surface in another manifold:

Proposition 3. *Let $F \subset B^4$ be a smoothly embedded nonorientable surface with odd b_1 bounding a knot $K \subset S^3$. Then there exists an orientable surface $F' \subset S^2 \times S^2 \setminus B^4$ which still bounds K , and has $b_1(F') = b_1(F) - 1$ and $e(F') = e(F) + 2$.*

The construction is similar to one in [Yas].

We then attach a -1 -framed 2-handle along K to get a four-manifold W , with boundary $S^3_{-1}(K)$. There is a closed, orientable surface Σ in W , formed by union of F' and the core of the 2-handle. By excizing a neighborhood of Σ from W , we get a negative semi-definite cobordism from a circle bundle over Σ to $S^3_{-1}(K)$. The definiteness of W gives us an inequality between the Heegaard-Floer d -invariants of its two boundaries, ultimately yielding:

Theorem 4. *Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then*

$$\frac{e(F)}{2} \leq 2d(S^3_{-1}(K)) + b_1(F).$$

That is, $\gamma_4(K, n) \geq n - 2d(S^3_{-1}(K))$.

Combining this theorem with the signature inequality yields Theorem 1, which can be written as

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - d(S^3_{-1}(K)).$$

The d -invariants of integer homology spheres are in general somewhat difficult to compute, though $d(S^3_{-1}(K))$ can in general be calculated from the filtered Heegaard Floer knot complex $CFK^\infty(K)$ [Pet]. When K admits a lens space surgery, however, these d -invariants can be read off from the Alexander polynomial of K . Using a recursive formula of Murasugi to calculate the signature of torus knots, we are able to prove Theorem 2, that $\gamma_4(T_{2k,2k-1}) \geq k - 1$. In fact, we can construct a surface $F_{2k,2k-1}$ bounding $T_{2k,2k-1}$ for which equality holds.

Proposition 5. *The torus knot $T_{2k,2k-1}$ has $\gamma_4(T_{2k,2k-1}) = k - 1$. That is, $T_{2k,2k-1}$ does not bound a punctured $\#^{k-2}\mathbb{RP}^2$ smoothly embedded in B^4 , and does bound a punctured $\#^{k-1}\mathbb{RP}^2$.*

The surface $F_{2k,2k-1}$ is an example of a more general construction. For each relatively prime p and q , we find a nonorientable surface $F_{p,q}$ in B^4 bounding $T_{p,q}$, whose first Betti number satisfies the recursion $b_1(F_{p,q}) = b_1(F_{p-2t,q-2h}) + 1$ where t and h are the minimal nonnegative representatives of $-q^{-1}$ modulo p and p^{-1} modulo q , respectively. We conjecture that these surfaces always have minimal genus, ie, that $b_1(F_{p,q}) = \gamma_4(T_{p,q})$.

In contrast to the orientable case, where the so-called Milnor conjecture $g_4(T_{p,q}) = g_3(T_{p,q})$ holds, we show that $\gamma_4(T_{4,3}) = 1$ while Teregaito has computed that $\gamma_3(T_{4,3}) = 2$ [Ter].

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2. CONSTRUCTING AN ORIENTABLE REPLACEMENT

In this section, we prove

Proposition (3). *Let $F \subset B^4$ be a smoothly embedded nonorientable surface with odd b_1 bounding a knot $K \subset S^3$. Then there exists an orientable surface $F' \subset S^2 \times S^2 \setminus B^4$ which still bounds K , and has $b_1(F') = b_1(F) - 1$ and $e(F') = e(F) + 2$.*

Proof. We break the proof into four steps.

Step 1: There is an embedded disk $D \subset B^4$, with boundary contained in F , such that $F \setminus \partial D$ is orientable.

Since F has odd b_1 , it is diffeomorphic to a punctured orientable surface boundary-connect summed with a Mobius band (Figure 2.1). Let $C \subset F$ be the core of the Mobius band; note that $F - C$ is orientable. After an ambient isotopy, we may arrange that C lies in the sphere of radius $1/2$, $S^3_{1/2} \subset B^4$, and that F intersects $S^3_{1/2}$ transversely. Think of C as a knot: it bounds some immersed disk D^2 in $S^3_{1/2}$, with clasp and ribbon singularities (Figure 2.2). We may remove the ribbon singularities by pushing the inner immersed segment in towards the centre of the 4-ball. To remove the clasp singularities, we push both immersed segments of the disk off the $1/2$ -level, one in towards the centre, and the other out towards the boundary. (The ability to push the surface both inwards and outwards is crucial, since a knot on the boundary of the B^4 bounds an embedded disk only if it is slice.) By a small isotopy, we may arrange that this embedded disk D bounding C intersects F transversely on its interior.

Let N be a small regular neighborhood of D .

Step 2: The intersection $\partial N \cap F$ is the link L shown in Figure 2.3.

N is diffeomorphic to $D \times D^2$, and intersects our surface F in a Mobius band (in the neighborhood of $\partial D = C$) and a collection of disks $pt \times D^2$ (neighborhoods of the transverse intersections of F with the interior of D). If we draw $S^3 = \partial N$ with its standard decomposition into solid tori $S^3 \cong S^1 \times D^2 \cup_{T^2} D^2 \times S^1$, we see $F \cap \partial N$ as the link L consisting of a $(2, 2k+1)$ -cable of the core of the first factor, together with a collection of l longitudes for the second. By construction, L bounds a Mobius band disjoint union a collection of l disks in $N \cong B^4$.

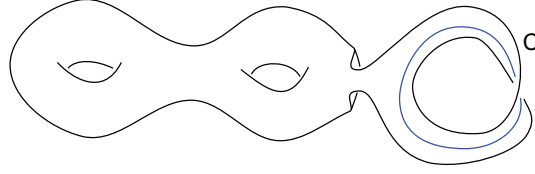
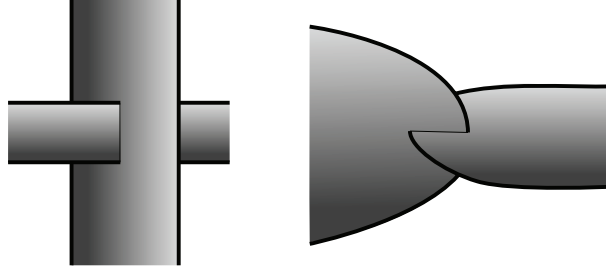
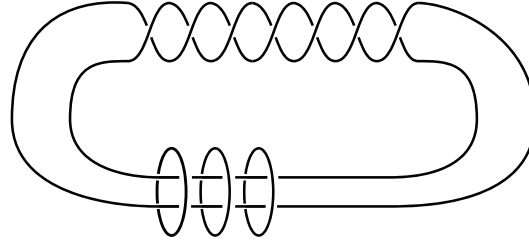
FIGURE 2.1. The surface F 

FIGURE 2.2. Ribbon and Clasp singularities

FIGURE 2.3. Our surface F intersects ∂N in a torus knot and an unlink

Step 3: L bounds $l + 1$ disjoint embedded disks in $S^2 \times S^2 \setminus B^4$

A handle decomposition for $S^2 \times S^2 \setminus B^4$ consists of two zero-framed 2-handles H_1 and H_2 attached along a Hopf link in the boundary S^3 , together with a 4-handle. To construct the slice disks for L , we begin with $|k| + l$ parallel copies of the core of H_2 and 2 parallel copies of the core of H_1 —their boundaries form a multi-Hopf link, with components $U_1, \dots, U_{|k|+l}, L_1, L_2$, as in the first frame of Figure 2.4. For each $1 \leq i \leq |k|$, connect U_i to L_1 with a twisted strip, and with one additional twisted strip, connect V_1 to V_2 . Call the surface consisting of the parallel cores and the strips E , and note that the boundary of E is isotopic to L . Since each strip connects a distinct disk to L_1 , E , or rather a slightly isotoped copy of E , is a collection of $l + 1$ disjoint embedded disks with boundary L .

Step 4: Construct F' , and compute $b_1(F')$ and $e(F')$.

If we excise N from B^4 , we are left with an orientable surface $F'' \subset S^3 \times [0, 1]$, with boundary K in $S^3 \times \{0\}$ and L in $S^3 \times \{1\}$. Attach $S^2 \times S^2 \setminus B^4$ along $S^3 \times \{1\}$ to form a larger manifold, still diffeomorphic to $S^2 \times S^2 \setminus B^4$. The slice disks E for L combine with F'' to form an orientable surface F' , whose only remaining boundary is the original knot K .

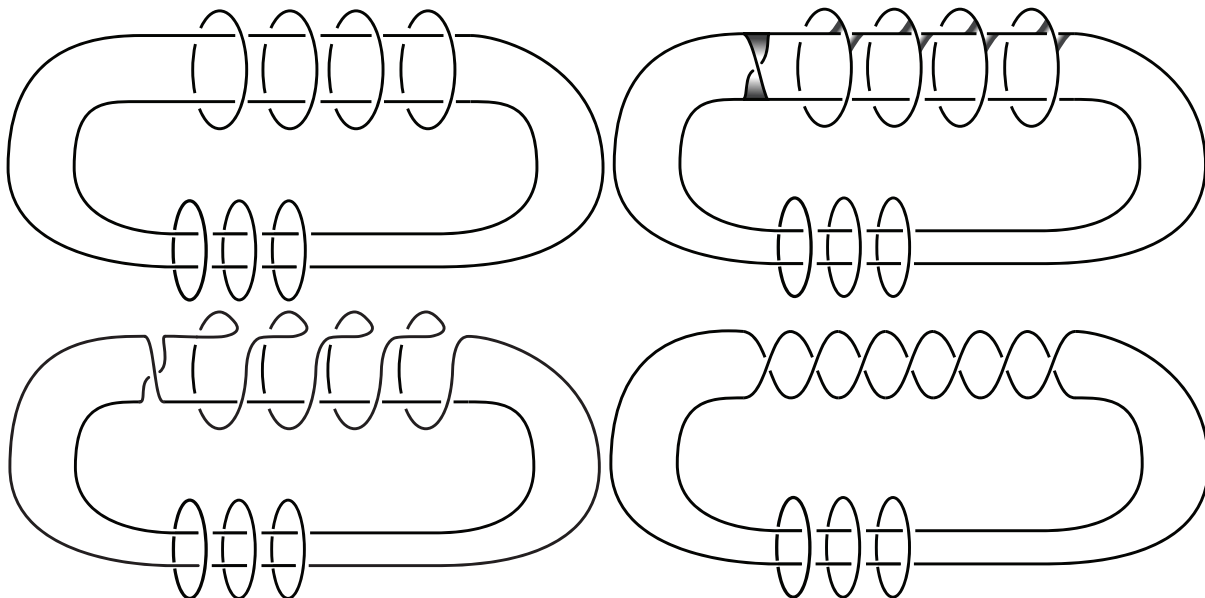


FIGURE 2.4. For $k = -7$, $l = 3$, we have drawn the multihopf link bounding a collection of parallel disks, the strips which join them to form E , and the boundary of E , which is isotopic to L .

Since we have removed l disks and an annulus from F , and replaced them with $l + 1$ disks, $b_1(F') = b_1(F) - 1$. It remains to compare the normal Euler numbers. The remaining l unknots, U_1, \dots, U_l have the same framing induced by E and $F \cap N$. The torus knot component of L is bounded by a Mobius band in $F \cap N$, and by an interesting disk in E . We invite the reader to verify that the induced framings differ by 2, due to the difference between the vertical twisted strip connecting V_1 to V_2 in E and the horizontal one in Mobius band. That is, $e(E) = e(F \cap N) + 2$. Since Euler number, like any self-intersection, is additive, $e(F') = e(F) + 2$. \square

For future reference, we note that the homology class $[F'] \in H_2(S^2 \times S^2 \setminus B^4)$ is $(2, m)$, in the basis given by H_1 and H_2 , with $m = |k| + l$. Since F' is orientable, its algebraic self-intersection number, $4m$, must be equal to its geometric self-intersection number, $e(F')$.

3. d -INVARIANTS

Heegaard Floer homology associates to a 3-manifold Y equipped with a Spin^c structure \mathfrak{t} a suite of $\mathbb{Z}[U]$ -modules which fit into a long exact sequence:

$$\cdots \rightarrow HF^-(Y, \mathfrak{t}) \xrightarrow{i} HF^\infty(Y, \mathfrak{t}) \xrightarrow{\pi} HF^+(Y, \mathfrak{t}) \xrightarrow{\delta} HF^-(Y, \mathfrak{t}) \rightarrow \cdots$$

If $c_1(\mathfrak{t})$ is torsion (in which case we also say that \mathfrak{t} is torsion), then there is a \mathbb{Q} -grading gr on each of these groups which is respected by i and π . The action of U decreases grading by 2. If Y is a rational homology sphere, then $HF^\infty(Y, \mathfrak{t}) \cong \mathbb{Z}[U, U^{-1}]$, and every Spin^c structure is torsion. In that case, the d -invariant (or correction term) $d(Y, \mathfrak{t})$ is the minimal grading of a non- \mathbb{Z} -torsion element of $HF^+(Y, \mathfrak{t})$ in the image of π .

If $b_1(Y) > 0$, then there is an additional action of $H := H_1(Y)/\text{Tors}$ on the HF groups, which decreases grading by 1. If for every torsion $\mathfrak{t} \in \text{Spin}^c(Y)$, $HF^\infty(Y, \mathfrak{t}) \cong \mathbb{Z}[U, U^{-1}] \otimes_{\mathbb{Z}} \Lambda^* H$, then we say that Y has standard HF^∞ . In that case, there are many correction terms, one for each generator of $\Lambda^* H$. We will be concerned with the bottom-most correction term, $d_b(Y, \mathfrak{t})$, defined to be the minimal grading of a nontorsion element of $HF^+(Y, \mathfrak{t})$ in the image of π and in the kernel of the H -action. The d -invariants terms will be useful to us because of their relationship to definite cobordisms.

Proposition 6. [OS1] *Let Y be a closed oriented 3-manifold (not necessarily connected) with standard HF^∞ , endowed with a torsion Spin^c structure \mathfrak{t} . If X is a negative semi-definite four-manifold bounding Y such that the restriction map $H^1(X; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z})$ is trivial, and \mathfrak{s} is a Spin^c structure on X restricting to \mathfrak{t} on Y , then*

$$c_1(\mathfrak{s})^2 + b_2^-(X) \leq 4d_b(Y, \mathfrak{t}) + 2b_1(Y).$$

In the previous section, we constructed an orientable surface $F' \subset S^2 \times S^2 \setminus B^4$ with boundary $K \subset S^3$. Attach a -1 -framed 2-handle along K to form a 4-manifold \bar{W} with boundary $S^3_{-1}(K)$ and intersection form

$$Q_{\bar{W}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We may cap off F' with the core of the 2-handle to form a closed surface Σ with genus $g = (b_1(F) - 1)/2$, homology class $(1, 2, m)$, and self-intersection

$$n := 4m - 1 = e(F) + 1 > 0$$

If we decompose $\bar{W} = \nu(\Sigma) \cup W$, then W will be a negative semi-definite cobordism from $Y_{g,n}$, the Euler number n circle bundle over Σ , to $S^3_{-1}(K)$. Alternatively, we can view W as a negative semi-definite four-manifold with disconnected boundary $Y_{g,-n} \amalg S^3_{-1}(K)$. To apply the above proposition, and so prove Theorem 4, we need to understand the homology, HF^∞ , and d -invariants of $Y_{g,n}$ and $S^3_{-1}(K)$, and the intersection form on W .

The Gysin sequence for the disk bundle $\nu(\Sigma)$ gives

$$0 \rightarrow H^1(\nu(\Sigma)) \rightarrow H^1(Y_{g,n}) \rightarrow H^2(\nu(\Sigma), Y_{g,n}) \xrightarrow{e} H^2(\nu(\Sigma)) \rightarrow H^2(Y_{g,n}) \rightarrow H^1(\Sigma) \rightarrow 0$$

where $e \in H^2(\nu(\Sigma)) \cong \mathbb{Z}$ is n times the generator. Thus $H^2(Y_{g,n}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n$. Note that the restriction of $H^1(\nu(\Sigma))$ to $H^1(Y_{g,n})$ is an isomorphism. Since $H^1(\bar{W}) = 0$ (no 1-handles were used in its construction), the Mayer-Vietoris sequence

$$0 \rightarrow H^1(\bar{W}) \rightarrow H^1(\nu(\Sigma)) \oplus H^1(W) \rightarrow H^1(Y_{g,n}) \rightarrow H^2(\bar{W}) \rightarrow H^2(\nu(\Sigma)) \oplus H^2(W) \rightarrow H^2(Y_{g,n})$$

shows that $H^1(W) = 0$, trivially satisfying the restriction hypothesis of Proposition 6. Since $H^2(\bar{W}) \cong \mathbb{Z}^3$ has no 2-torsion, a Spin^c structure on \bar{W} is determined by its first chern class. Any Spin^c structure on W will give us some inequality between d -invariants, but we will only need to consider a certain Spin^c structure \mathfrak{s}_t with $PD(c_1(\mathfrak{s}_t)) = (\pm 1, 2, 2a)$, where

$$a = \frac{2(m - g) - 1 \pm 1}{4}$$

and the sign is chosen so as to make a an integer. The given vector is characteristic for $Q_{\bar{W}}$, so does correspond to a Spin^c structure. Crucially for our later use, $c_1(\mathfrak{s}_t)$ evaluates to $n - 2g$ on Σ .

To compute the c_1^2 term in the proposition, we decompose the intersection form of \overline{W} in terms of the \mathbb{Q} -valued intersection forms on $\nu(\Sigma)$ and W : if $c \in H^2(\overline{W})$, then

$$Q_W(c) = Q_{\nu(\Sigma)}(c|_{\nu(\Sigma)}) + Q_W(c|_W).$$

A generator of $H^2(\nu(\Sigma), Y_{g,n})$ maps to n times the generator of $H^2(\nu(\Sigma))$ in the gysin sequence above, so $Q_{\nu(\Sigma)} = (\frac{1}{n})$. The value of $c|_{\nu(\Sigma)} \in H^2(\nu(\Sigma))$ is determined by integrating it over Σ , giving

$$(3.1) \quad Q_{\overline{W}}(c) = \frac{\langle c, [\Sigma] \rangle^2}{n} + Q_W(c|_W).$$

In our case,

$$c_1(\mathfrak{s}_t|_W)^2 = Q_{\overline{W}}(c_1(\mathfrak{s}_t)) - \frac{\langle c_1(\mathfrak{s}_t), [\Sigma] \rangle^2}{n} = -1 + 8a - \frac{(n-2g)^2}{n} = -2 \pm 2 - \frac{4g^2}{n}.$$

The relevant d -invariant of $Y_{g,-n}$ is computed in section 9 of [OS1], for use in their proof of the Thom conjecture. If $n > 2g$, then

$$d_b(Y_{g,-n}, \mathfrak{s}_t|_{Y_{g,-n}}) = \frac{1}{4} - \frac{g^2}{n} - \frac{n}{4}.$$

That calculation uses the integer surgeries exact sequence associated to the Borromean knot in $K \subset \#^{2g} S^1 \times S^2$: the $-n$ surgery on K gives $Y_{g,-n}$. Since $\#^{2g} S^1 \times S^2$ has standard HF^∞ , so does $Y_{g,-n}$ (cf. Proposition 9.4 of [OS1]). Finally, since $S^3_{-1}(K)$ is an integer homology sphere, it also has standard HF^∞ .

We are now ready to prove Theorem 4. By Proposition 6, we have

$$c_1(\mathfrak{s}_t)^2 + b_2^-(W) \leq 4d_b(Y_{g,-n}, \mathfrak{t}) + 4d(S^3_{-1}(K)) + 2b_1(Y_{g,-n}) + 2b_1(S^3_{-1}(K)).$$

After substituting all the values computed above, this reduces to

$$\left(-2 \pm 2 - \frac{4g^2}{n}\right) + 2 \leq 4\left(\frac{1}{4} - \frac{g^2}{n} - \frac{n}{4}\right) + 4d(S^3_{-1}(K)) + 2(2g).$$

If we take the unfavorable sign on ± 2 , and recall that $b_1(F) = 2g + 1$ and $e(F) + 1 = n$, we get the inequality

$$(3.2) \quad \frac{e(F)}{2} \leq 2d(S^3_{-1}(K)) + b_1(F).$$

This argument relied on a value for $d_b(Y_{g,-n})$ only valid if $n > 2g$, ie, $e(F) + 2 \geq b_1(F)$. Proposition 6, applied to the surgery cobordism $S^3 \rightarrow S^3_{-1}(K)$, guarantees that $d(S^3_{-1}(K)) \geq 0$, so if $e(F) + 2 < b_1(F)$, the above inequality is trivially satisfied.

The initial construction of an orientable replacement required that $b_1(F)$ be odd. Luckily, both sides of Equation 3.2 change by the same amount under a positive real 'blow-up.' More precisely, if we connect sum $F \subset B^4$ with the standard embedding of $\mathbb{RP}^2 \subset S^4$ with Euler number $+2$, then both b_1 and $e/2$ increase by 1. One way to construct this \mathbb{RP}^2 is to glue together the Mobius band and disk bounding $T_{2,-1}$ (cf the mirror of Figure 1.1 at $k = 0$), then push them off into opposite sides of $S^3 \subset S^4$. If $b_1(F)$ is even, we may apply Equation 3.2 to $F \# \mathbb{RP}^2$, and so deduce it for F .

This completes the proof of Theorem 4, and hence of Theorem 1.

Remark. Our final lower bound on γ_4 is the gap $\frac{\sigma(K)}{2} - d(S^3_{-1}(K))$. For alternating knots, this quantity is nonpositive—in [OS2], Ozsváth and Szabó show that

$$d(S^3_{-1}(K)) = \max\left(0, 2 \left\lceil \frac{\sigma(K)}{4} \right\rceil\right)$$

For nonalternating knots, $\frac{\sigma(-)}{2}$ and $d(S_{-1}^3(-))$ can diverge widely, though both invariants satisfy a crossing-change inequality [Pet]:

$$\eta(K_+) \leq \eta(K_-) \leq \eta(K_+) + 2.$$

If K becomes alternating after c crossing changes, then $\frac{\sigma(K)}{2} - d(S_{-1}^3(K))$ can be as large as $2c$.

4. TORUS KNOTS

Signatures of torus knots satisfy a recursion relation [MK]. If $\sigma(p, q) := \sigma(T_{-p, q})$, then

$$\sigma(p, q) = \begin{cases} \sigma(q, p) & \text{if } q > p \\ \sigma(p - 2q, q) + q^2 (-1) & \text{if } 2q < p \text{ (} q \text{ odd)} \\ -\sigma(2q - p, p) + q^2 - 2 (+1) & \text{if } 2q > p \text{ (} q \text{ odd)} \\ p - 1 & \text{if } q = 2 \\ 0 & \text{if } q = 1 \end{cases}$$

Let $\sigma_k := \sigma(T_{-2k, 2k-1}) = \sigma(2k, 2k-1)$. Applying the first and third conditions twice, we arrive at the recursion

$$\sigma_k = 4k - 2 + \sigma_{k-1},$$

whence $\sigma_k = 2k^2 - 2$.

The d -invariants of torus knots are also simple to compute, since they admit lens space surgeries.

Proposition 7. [OS1] *Let K be a knot admitting a positive lens space surgery. Then*

$$d_{-1/2}(S_0^3(K)) = -\frac{1}{2} \quad \text{and} \quad d_{1/2}(S_0^3(K)) = \frac{1}{2} - 2t_0$$

where if

$$\Delta_K(T) = a_0 + \sum_{j=1}^d a_j (T^j + T^{-j})$$

then

$$t_0 = \sum_{j=1}^d j a_j.$$

The d -invariants of zero-surgery are related to those of ± 1 -surgery via Proposition 4.12 of [OS1]:

$$d(S_{-1}^3(K)) = d_{-1/2}(S_0^3(K)) + \frac{1}{2} \quad d(S_1^3(K)) = d_{1/2}(S_0^3(K)) - \frac{1}{2}.$$

Since $T_{p, q}$ admits a positive lens space surgery, we have

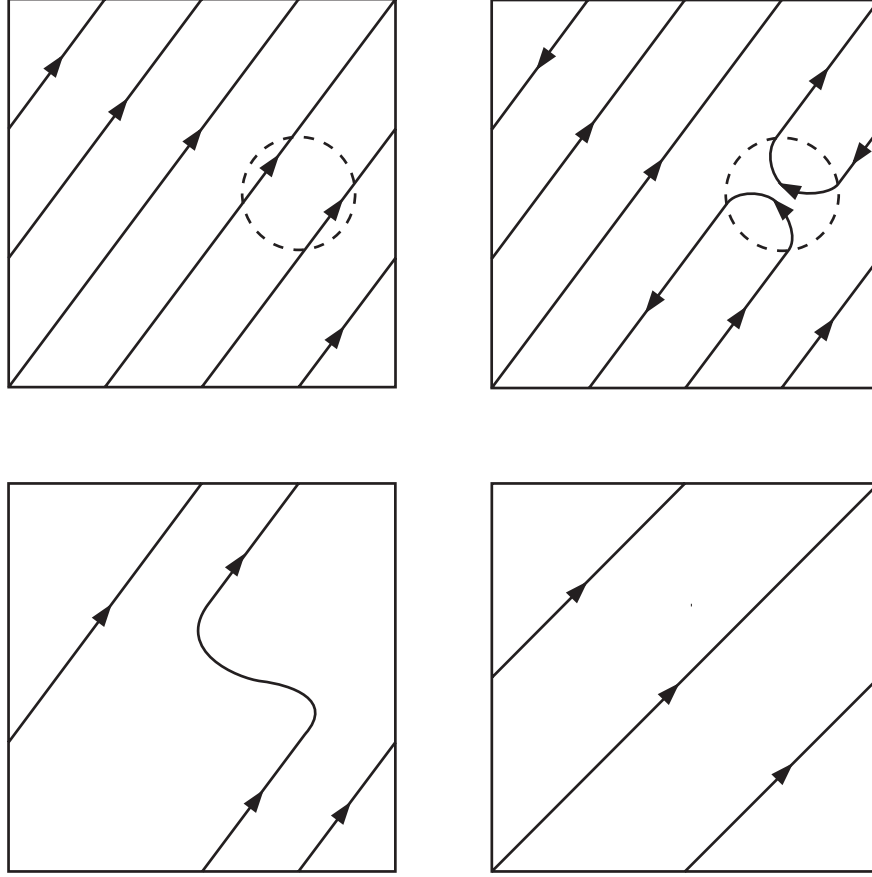
$$d(S_{-1}^3(T_{-p, q})) = -d(S_1^3(T_{p, q})) = -\left(d_{1/2}(S_0^3(T_{p, q})) - \frac{1}{2}\right) = 2t_0.$$

The Alexander polynomial of $T_{p, q}$ is

$$\Delta_{T_{p, q}}(T) = T^{-(p-1)(q-1)/2} \frac{(1-T)(1-T^{pq})}{(1-T^p)(1-T^q)}.$$

For torus knots $T_{2k, 2k-1}$, the Alexander polynomial has a simple form:

$$\Delta_{T_{2k, 2k-1}} = \sum_{j=1}^{k-1} T^{j(2k-1)} - T^{j(2k-1)-(k-j)} + T^{-j(2k-1)} - T^{-j(2k-1)+(k-j)}$$

FIGURE 4.1. A cobordism from $T_{4,3}$ to $T_{2,1}$

so

$$t_0 = \sum_{j=1}^{k-1} j(2k-1) - (j(2k-1) - (k-j)) = \sum_{j=1}^{k-1} k-j = \frac{k^2-k}{2}.$$

Hence

$$d(S_{-1}^3(T_{-2k,2k-1})) = k^2 - k.$$

The relevant difference between signature and d is

$$\frac{\sigma}{2} - d = k^2 - 1 - (k^2 - k) = k - 1.$$

Of course, the reflection of a surface bounding $T_{2k,2k-1}$ bounds $T_{-2k,2k-1}$.

Corollary 8. *If $F \subset B^4$ is a smoothly embedded nonorientable surface bounding $T_{2k,2k-1} \subset S^3$, then $b_1(F) \geq k - 1$.*

We obtain this lower bound by the following construction. Consider $T_{p,q}$ as actually lying in a standard torus, as in Figure 4.1. Take any two adjacent strands and join them with a strip, or, equivalently, perform

an index 1 Morse move merging them. The resulting cobordism is nonorientable, since the strands were parallel; it is a punctured Mobius band. Since the resulting knot still lives on the torus, it must be $T_{r,s}$ for some r and s . The values of r and s can be easily computed by orienting the resulting knot and counting the signed intersection with the horizontal and vertical generators of $H_1(T^2)$. A short calculation shows that

$$r = p - 2t \quad s = q - 2h$$

where $t \equiv -q^{-1} \pmod p$, with $0 \leq t < p$, and $h \equiv p^{-1} \pmod q$, with $0 \leq h < q$. After an isotopy, $T_{r,s}$ will be in standard, taut form on the torus, and we can repeat the process. Eventually, we arrive at $T_{n,1}$ for some n , which is just an unknot. By concatenating all of these cobordisms, then capping off the final unknot with a disk, we have successfully found a surface $F_{p,q}$ in B^4 bounding $T_{p,q}$.

For example, if $p = 2k$ and $q = 2k - 1$, we have $t = -(-1)^{-1} = 1$ and $h = 1^{-1} = 1$, giving $r = 2k - 2$ and $s = 2k - 3$. Thus $T_{2k,2k-1}$ has a $\chi = -1$ cobordism to $T_{2(k-1),2(k-1)-1}$. Concatenate $k - 1$ of these, then cap off $T_{2,1}$ with a disk to get a closed surface $F_{2k,2k-1} \subset B^4$ bounding $T_{2k,2k-1}$, with $b_1(F_{2k,2k-1}) = k - 1$.

Since the isotopies and Morse moves take place inside of the torus, we can actually embed each of these cobordisms in a thickened torus $T^2 \times [-\epsilon, \epsilon]$ in S^3 , where we view the $[-\epsilon, \epsilon]$ direction as a sort of time. The obstruction to embedding all of $F_{p,q}$ in S^3 is that the final disk bounding $T_{n,1}$ cuts through all of the previous layers, unless $n = 0$. To get a surface in S^3 , we must continue with these within-torus cobordisms: $T_{n,1} \mapsto T_{n-2,1} \mapsto \dots$. If n is even, or, equivalently, if pq was even to start, then we do get a surface in S^3 . Teragaito has computed $\gamma_3(T_{p,q})$, and it agrees with $b_1(F)$ [Ter]. For example, $\gamma_3(T_{2k,2k-1}) = k$. If n is odd, then this construction fails to give a surface in S^3 , though a slight modification (cf. [Ter] Remark 4.9) will do.

We conjecture that the surfaces $F_{p,q}$ bounding $T_{p,q}$ are best possible, that $b_1(F_{p,q}) = \gamma_4(T_{p,q})$. Many pairs (p, q) for which the conjecture holds can be certified using the d -invariant bounds of this paper. Similar invariants, derived by considering larger surgeries on the knot, give even more examples. These stronger bounds will be discussed in a forthcoming paper.

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